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Comments on "Study of Nonlinear Systems"

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A FEW points of the recent comments by Rao,[1] have been advanced in the light of several published materials.

It is a new contribution to apply the direct interchange between dependent and independent variables. Nonlinear, ordinary differential equations, especially Riccati equations, have been actively discussed among radio engineers for nonuniform transmission lines where the line parameters vary continuously along the line length. However, a preliminary literature search seems to exclude the direct interchange between dependent and independent variables.

One of the key points in obtaining closed form solutions by factorizations in Ref. 1 has been the following relationship:

$$(d/dx)[F(x)/f(x)] = \lambda f(x) \qquad (1)$$

This is a Bernoulli nonlinear differential equation for $f(x)$ when $F(x)$ and λ are specified,

$$df(x)/dx = f(x)\ (d/dx)[\ln F(x)] - [\lambda/F(x)]f^3(x) \qquad (2)$$

This interrelationship between $f(x)$ and $F(x)$ is restricting a possible wider application of Rao's method. In the process of generalizing the original Konyukov's nonlinear differential equation[2,3] a similar relation has been derived for variable coefficients.[4] It is also noted that the book by Murphy[5] goes into a detailed discussion on Abel's nonlinear differential equations, including the separable case of Eq. (1).

At this point, Rao's method has been used for a generalized Konyukov's nonlinear differential equation. The direct interchange of dependent and independent variables has been given to

$$xd^2x/dt^2 + P(t)(dx/dt)^2 + Q(t)x^3 + R(t)x^2 = 0 \qquad (3)$$

where $P(t)$, $Q(t)$, and $R(t)$ are arbitrary functions of the independent variable t. Then, the result is

$$d^2t/dx^2 = [P(t)/x]\, dt/dx + [Q(t)x^2 + R(t)x](dt/dx)^3 \qquad (4)$$

Evidently this appears not solvable for general $P(t)$, $Q(t)$, and $R(t)$. In contrast, the published materials show how to obtain closed forms for the following cases:

$P(t) = -2$, $Q(t) =$ constant, $R(t) =$ constant, Ref. 2

$P(t) = -2$, $Q(t) =$ arbitrary, $R(t) =$ constant, Ref. 3

$P(t) = -2$, $Q(t) =$ arbitrary, $R(t) =$ a solution of Bernoulli's equation Ref. 4

Equation (4) is solvable if $P(t)$, $Q(t)$, and $R(t)$ are all constant, since it then becomes a Bernoulli's nonlinear differential equation for dt/dx. Thus, it appears that another key point of why Rao's method works successfully on Eq. (2) of Ref. 1 is that it contains no variable coefficients of t.

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Received September 17, 1970.

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Comment on "Unsymmetrical Bending of Shells of Revolution"

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IN Ref. 1, Blech has presented a set of eight first-order ordinary differential equations for the bending analysis of unsymmetrically loaded shells of revolution on the basis of Sander's first-order shell theory. This formulation was first suggested by Goldberg et al.[2] and has been successfully employed by several authors for the equilibrium, stability and vibration problems of shells.

Received August 31, 1970.

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Table 1 Modifications of equations

Order of equation in Ref. 1 [Eqs. (1)]	Multiplier
1	-1
4	-1
6	ρ
7	ρ
5	ρ
8	$\lambda^2\rho$
2	-1
3	-1

For the solution of the resulting system of equations, in addition to the numerical integration technique (references cited in Ref. 1), matrix progression[3] and modified finite-difference methods[4] have been efficiently used.

The purpose of this comment is twofold. First, to recast the governing equations [Eqs. (1)] into a simpler form with a symmetric coefficient matrix (A) which is particularly useful when finite-difference method is used for the solution, and second, to point out a number of misprints in Eqs. (3a–3x).

The modified form of the equation is obtained by choosing the fundamental unknowns $\{Z\}$ as follows

$$Z_1 = \rho t_\xi, \quad Z_2 = \lambda^2 \rho m_\xi, \quad Z_3 = u_\theta, \quad Z_4 = W$$

$$Z_5 = u_\xi, \quad Z_6 = \phi_\xi, \quad Z_7 = \rho \hat{t}_{\xi\theta}, \quad Z_8 = \rho \hat{f}_\xi$$

and then multiplying each equation by the coefficient shown in Table 1 and rewriting the resulting equations in the order shown in Table 1.

The modified form of the equations

$$BZ' - AZ = P$$

where

$$B = \left[\begin{array}{c|c} 0 & I_1 \\ \hline -I_1 & 0 \end{array}\right]$$

I_1 being diagonal submatrix and has the following form

$$[I_1] = \begin{bmatrix} -1 & & 0 & \\ & -1 & & \\ 0 & 1 & & \\ & & & 1 \end{bmatrix}$$

If the nodal values of the fundamental unknowns are arranged in the order just indicated above, a symmetric coefficient matrix for the finite-difference equations can be obtained.

It should be noted that a symmetric coefficient matrix can still be obtained for laminated orthotropic shells of revolution even when account is taken of the shear deformation (in which case the governing equations are ten first-order ordinary differential equations,[5] provided that the thermomechanical characteristics of the shell are independent of θ).

The following misprints in Ref. (1) have been noted; the correct forms are:

$$A_{11} = A_{44} = \tag{3a}$$

$$A_{12} = -A_{65} = \tag{3b}$$

$$A_{13} = -A_{75} = \tag{3c}$$

$$A_{15} = 1/b \tag{3d}$$

$$A_{21} = -A_{56} = \tag{3e}$$

$$A_{22} = -A_{77} = -\tfrac{1}{2}A_{66} = \tag{3f}$$

$$A_{23} = \gamma A_{24} = -A_{76} = -\lambda^2\gamma A_{86} = \gamma(\lambda^2 d/bc)(n/\rho)(3\omega_\theta - \omega_\xi) \tag{3g}$$

$$A_{26} = 2/\{bc(1-\nu)\} \tag{3h}$$

$$A_{31} = -A_{57} \tag{3i}$$

$$A_{42} = -(1/\lambda^2)A_{68} \tag{3k}$$

$$A_{43} = -(1/\lambda^2)A_{78} \tag{3l}$$

$$A_{48} = 1/d \tag{3m}$$

$$-A_{54} = \tag{3q}$$

$$A_{55} = A_{88} = \tag{3w}$$

$$A_{87} = \tag{3x}$$

The two coefficients A_{63} and A_{72} are nonzero and are given by

$$A_{63} = A_{72} = b(1-\nu^2)(n/\rho)\omega_\theta[1 + (\lambda^2 d/b)\, n^2/\rho^2]$$

In addition, all the temperature terms in Eqs. (2b) should have a minus sign.

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Reply by Author to A. K. Noor

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The author wishes to thank A. Noor on his comment. As to the choice of the fundamental unknowns, the author found that the system of equations as presented in his Note could be adequately solved by standard differential equations solver routines which are available nowadays as part of a library of subroutines of digital computers. In particular, the author used a Runge-Kutta fourth-order routine for the solution of the equations.

Concerning the second portion of the comment. The misprints which are pointed out were checked against the author's original notes. The author concurs with the corrections.

It seems that Eq. (3q) should read:

$$-A_{54} = -\lambda^2 A_{81} =$$

Received October 13, 1970.

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